Finsler Geometric *Local* Indicator of Chaos for single orbits in the Hénon–Heiles hamiltonian.

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(June 15, 1998)

Translating the dynamics of the Hénon–Heiles (HH) hamiltonian as a geodesic flow on a *Finsler manifold*, we obtain a *local and synthetic* Geometric Indicator of Chaos (GIC) for two degrees of freedom (dof) continuous dynamical systems (DS's). It represents a link between *local* quantities and *asymptotic behaviour* of orbits and gives a strikingly evident, *one-to-one*, correspondence between geometry and instability. Going beyond the results attained using the customary dynamical approach and improving also on the *global* criteria established within the Riemannian framework, the GIC is able to discriminate between regular and stochastic orbits on a given energy surface, simply on the basis of the value it assumes along a relatively small piece of the trajectory, *without long integrations of the dynamics* and without *any reference to a perturbation*.

PACS: 05.45.+b ; 02.40.-k ; 02.90.+p

The presence of instability in DS's is usually recognized *a posteriori*, looking at the long time evolution of *small* perturbations, whose average exponential growth is interpreted as *the* signature of Chaos. Skipping here most of the issues related to the universal meaning attributed to it, we remark how the procedures and tools having at the grounds such a criterion alone manifest their limitations whenever is investigated the behaviour of DS's at the *boundary* between *quasi-integrability* and *stochasticity* (e.g., [1,2]). Instead, the search for *a priori, synthetic* signatures of Chaos, dates back to Toda [3] and continued, across interesting investigations on the mechanisms of transition to stochasticity (see e.g. [4]), up to the recently revived Riemannian *Geometrodynamical approach* (GDA) [5,6]. Though most of its results relate to high dimensional hamiltonian systems (for which some approximations are justified by the large number of dof or some *weak form* of the ergodic hypotesis), more recently, this approach has been tested also for *small* DS's, giving outcomes clearly supporting its reliability, [7,2]. Nevertheless, if in the case of large DS's considered the agreement between the GDA and the *customary* tools used to detect Chaos has revealed to be rather satisfactory, at least as long as the approximations are well justified [8], some discrepancies emerge in the case of few dof systems [2,9]. For the latter, the GDA provided an alternative way to recover *most* of the results obtained with the tangent dynamics equations, suggesting deeper hints for the understanding of the sources of Chaos and giving in addition some *global* criteria to single out a transition in the overall behaviour [7]. However, this criterion is unable to correctly detect the occurrence of Chaos in single orbits [10], as it renounces, *in principle*, to *intrinsically* describe the behaviour of *individual trajectories*. Within the Riemannian approach to few dof systems, this issue has been addressed, up to now, only resorting to a numerical procedure analogous to the integration of the tangent dynamics equations, whose results have been *generally* confirmed (though not always). Recently Kandrup [11] investigated in details the relationships existing between *local dynamical behaviour* and *local geometric features* of the Jacobi (Riemannian) manifold for some two-dimensional DS's, obtaining qualitative correlations among average curvature and its fluctuations and somewhat more ambiguous ones between curvature fluctuations and *short time Lyapunov exponents*. In summary, the Riemannian GDA has been able, up to now, either to *intrinsically* describe the *average behaviour* of a DS or to single out the *individual* orbits instability *a posteriori*, as in the Hamiltonian description. Both these approaches have revealed unable to find an *intrinsic and individual* indicator of *long-term behaviour of orbits* as the GIC here presented, built within the Finsler GDA and which *cannot even to be defined*, (for the HH system) within the other above mentioned frameworks. Notwithstanding its *local* character (in both spatial and temporal meanings), it reveals to be unambiguously related to Lyapunov Characteristic Numbers (LCN's), *i.e.*, to asymptotic quantities, usually computed with reference to a perturbation! We claim then that it represents a strong indication (if not a proof) that the GDA is able not only to *reproduce* and to *explain* the results obtained with the usual tools, but even to go beyond them.

One of the main recent results of the GDA is the confirmation that the onset of unpredictability in the geometric transcription of realistic (large) DS's is driven by the mechanism of parametric instability [5,6] and thus differs completely from what occurs in the geodesic flows of abstract Ergodic theory. Indeed, most of phase space of large *physical* DS's is not characterized by (constant) negative curvatures, but stochasticity is caused by the *quasi random* fluctuations of (mostly) positive curvatures. For few dimensional DS's however, such random character cannot be assumed and instead is *parametric resonance*, similar to that occurring in the Mathieu equation, to bring about instability, [6,7,12]. Nevertheless, we find

below that even for two dof systems such a mechanism cannot be singled out in a *naïve* way and very intricate combinations of geometric features of the manifold are linked to instability in a non trivial way.

We already discussed the motivations for an extension of the GDA to include non Riemannian manifolds [12] and we also pointed out its somewhat *greater effectiveness* with respect to the *usual* tools in the computations of *instability exponents* [2,13]. We refer to [5,6,8] for a detailed description of the GDA, to [2,12,14] for its implementation within Finsler manifolds and to [9] for a thorough discussion of most of the points here only sketched.

Within the GDA the trajectories of a $\mathcal{N}$ dof system become the geodesics of suitable differential manifolds, which, in Finsler geometry, are $(\mathcal{N}+1)-$dimensional and represent a generalization of Riemannian ones. The stability of the flow is determined by the *Jacobi–Levi-Civita* (JLC) equations of geodesic spread:

$$\frac{\nabla}{ds}\left(\frac{\nabla z^a}{ds}\right) \ + \ \mathcal{H}^a{}_c \, z^c \ = \ 0 \ , \quad (a = 0, 1, \ldots, \mathcal{N}) \ , \quad (1)$$

being z^a the perturbation, ∇/ds the covariant derivative *along the geodesic* and the *stability tensor*, $\boldsymbol{\mathcal{H}}$, [6,8], derives from the (generalized) curvature tensor of the manifold. The Finsler *time-parameter* $s \equiv s_F$ is defined through the Lagrangian function $\mathcal{L}$, as $ds_F = \mathcal{L}\, dt$ and possesses a built-in invariance with respect to an arbitrary rescaling of the *Newtonian* time t, [15]. The *local* behaviour of geodesics is determined by the eigenvalues of $\boldsymbol{\mathcal{H}}$, which are the *principal sectional curvatures* (psc's) defined *by the given geodesic* on the manifold [8]. For a $\mathcal{N}$ degrees of freedom DS, once a geodesic is chosen, the Finsler stability tensor possesses $(\mathcal{N}+1)$ eigenvalues, $(\{\lambda_A\}, \ A = 0, 1, \ldots, \mathcal{N})$ one of which vanishes identically, $\lambda_0 \equiv 0$, associated with a *neutral* eigenvector, along the tangent to the geodesic. In the case of a *standard* hamiltonian system (*i.e.* without gyroscopic terms), $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\, a_{ij} p^i p^j + \mathcal{U}(\mathbf{q})$ we have then, [2,12]:

$$\lambda_i = t'(B + t'\mu_i) \ , \quad (i = 1, \ldots, \mathcal{N}) \qquad (2)$$

where B is related to the time derivatives of the Lagrangian, the $\{\mu_i\}$ are the eigenvalues of the hessian $\mathcal{U}_{,ij}\,(\mathbf{q})$ and the prime denotes the derivative wrt to s_F. In this letter we will deal with the well known two dimensional HH system, whose Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(x^2 + y^2 + p_x^2 + p_y^2\right) + x^2 y - \frac{y^3}{3}. \qquad (3)$$

We find that for this DS (as well as for generic realistic ones, in spite of some persistent claims, e.g. [16]) negative curvatures are quite unable to explain the asymptotic character of orbits. The mechanism of *parametric instability*, due to fluctuations of usually positive curvatures [5,6] manifests *naïvely* however only for some many dof systems, being instead hardly perceived in this case. For example, the analysis of spectra, [9], shows how intermingled and far from trivial are the relationships with the *elementary* theory of *Mathieu-like* equations. Such seemingly discouraging inconsistencies driven us to check for different signatures of instability. The Finslerian approach allows to consider, even for two dof Hamiltonians, the possible anisotropy of the manifold, which results to play a crucial role, via the *Schur theorem* [8] in the mechanism of instability: fluctuating curvatures require also that the manifold is anisotropic. The connection between curvatures variations along a geodesic and anisotropy, on one side, and growth and *rotation* of perturbation, on the other, is far from being clear and is currently investigated: how the former can interact to steer the latter can at the moment only be guessed. For a two dof system, the associated Finsler manifold is three dimensional and its curvature properties *along a given geodesic* are described by the two non vanishing psc's, $\lambda_{1,2}$, which are invariant functions on the tangent space, representing the sectional (*i.e.* gaussian) curvatures in the two-planes defined from the flow and the two (non trivial) eigenvectors of $\boldsymbol{\mathcal{H}}$. Given them, we can characterize the way the geodesic explores the manifold through the (half) *Ricci curvature* (along the flow) and the *anisotropy*, $\kappa[\mathbf{q}(s), \mathbf{p}(s)]$ and $\vartheta[\mathbf{q}(s), \mathbf{p}(s)]$, respectively:

$$\kappa \doteq \frac{\lambda_1 + \lambda_2}{2} \equiv \frac{\mathrm{Tr}(\boldsymbol{\mathcal{H}})}{2} = \frac{\mathsf{Ric_F}(\mathbf{u})}{2} \quad ; \quad \vartheta \doteq \frac{\lambda_1 - \lambda_2}{2} \ . \qquad (4)$$

An exhaustive statistical analysis of the behaviour of κ and ϑ (or equivalently of $\lambda_{1,2}$) and the details of the logical path leading to the *synthetic indicator* are presented elsewhere, [9]. We found that along a generic geodesic the $\{\lambda_i\}$ oscillate around their average values, the fluctuations of Ricci curvature in general turns out to be however much smaller than those of the sectional ones, which are indeed almost anti-correlated. Such an effect is particularly evident in the HH case, as $\Delta\mathcal{U} \equiv 2$. So, the manifold appear to be everywhere anisotropic but with psc's always (λ_1) or mostly (λ_2) positive. Fluctuations (and then anisotropy) increase with energy, showing a *global* qualitative change in correspondence of appearance of stochasticity, [9]. Although smaller, the overall fluctuations of Ricci curvature seem nevertheless to influence appreciably the stability of geodesics. Moreover, Schur theorem asserts that the two quantities must be related. We then look for a relative measure of anisotropy fluctuations compared to overall curvature variations. Correlations between two quantities $A(s)$ and $B(s)$ reflect in the functional:

$$\widetilde{\mathcal{C}}_S[A, B] \ \doteq \ \frac{\langle A \cdot B \rangle_S}{\left(\langle A^2 \rangle_S \ \cdot \ \langle B^2 \rangle_S\right)^{1/2}} \ , \qquad (5)$$

which clearly depends on the averaging interval S, dependence understood in the sequel. An indication about the

relative importance of anisotropy wrt average curvature fluctuations can be obtained comparing the phase space normalized correlation functions $\widetilde{\mathcal{C}}[\vartheta, \delta\vartheta]$ and $\widetilde{\mathcal{C}}[\kappa, \delta\kappa]$, being $\delta A(s) \doteq A(s) - \langle A \rangle$. Using them we build up a quantity which apparently describes only the *local* geometric features of the sub-manifold explored (in a lapse S) by the geodesic having initial conditions $[\mathbf{q}(0), \mathbf{p}(0)] = (\mathbf{q}_0, \mathbf{p}_0)$:

$$R_F[S] \;\equiv\; R_F(\mathbf{q}_0, \mathbf{p}_0) \;\doteq\; \frac{\widetilde{\mathcal{C}}[\vartheta, \delta\vartheta]}{\widetilde{\mathcal{C}}[\kappa, \delta\kappa]} \;\geq\; 0 \tag{6}$$

However, the inspection of figure 1 shows a striking correspondence between the PSS's and the map of R_F^2. Indeed, figures 1 a) and b), which refer to the *typical* energy value $E = 1/8$, show that smaller is the value of $R_F(\mathbf{q}_0, \mathbf{p}_0)$, *more regular* is the geodesic passing through $(\mathbf{q}_0, \mathbf{p}_0)$. To obtain the plot of figure 1 b), we chosen a grid of points $\{y_0, p_{y_0}\}$ on the PSS $x = 0$, chosing p_{x_0} such that $H = E$ and then numerically integrated the geodesic equations, computing the correlation functions entering $R_F(\mathbf{q}_0, \mathbf{p}_0)$. The same results have been obtained at all the energies; the more *essential*, though less *suggestive*, histograms of figure 2, confirm that our GCI is able to depict correctly the single orbits behaviour up to the dissociation energy. The columns in this plot represent the values of $R_{F_E}^2$, defined as

$$R_{F_E}^2(\mathbf{q}_0, \mathbf{p}_0) \;\doteq\; \alpha^2(E)\; R_F^2(\mathbf{q}_0, \mathbf{p}_0)\;, \tag{7}$$

where $\alpha(E) = \alpha_1 \cdot E^2$ is a scaling factor to get rids of a *global* energy dependence, useful to compare results at different energies. For each energy, the $R_{F_E}^2$ values are reported for a sample of seven initial conditions, chosen among those *topologically equivalent* to the ones indicated in figure 1b) for $E = 1/8$. We observe that while chaotic trajectories are always characterized by $R_{F_E}^2$ values around its upper limit (normalized to unity, this is because we use $R_{F_E}^2$ instead of R_F), regular orbits have instead considerably smaller values, which is however higher for those geodesics *tending to become chaotic earlier*, as the energy increases. In particular, from both the figures, we see that the geodesics in the *regular islands* located on the p_y-axis of the PSS are recognized by our GIC as *nearer* to chaotic ones than those belonging either to the large regular island on the $y-$axis or to the *banana* region; this explains why these islands *disappear* earlier as the energy increases and gives also some insights on the causes of the (partial) failure of the Riemannian approach to describe these *peculiar* orbits [2,7,10]. Moreover, very interesting insights can be obtained looking at the *relaxation patterns* of R_F as function of the averaging time S [9], as can be perceived from the *diffusive* behaviour around the border of regular islands. Well then, a geometric *local* quantity turns out to be deeply related to the *asymptotic* behaviour of geodesics. We stress that the values plotted in the map and histograms are obtained through computations of correlation functions over time intervals much shorter than those needed to obtain either the PSS's or the LCN's: it suffices to follow a geodesic for as few as fifteen periods in order to see if the value of R_F attains the upper limit characterising the *stochastic sea* or tends to a lower value, which though different for distinct regular orbits, is however always smaller than for chaotic ones. A *local signature* of asymptotic instability acquires a special significance also on the light of the issue of reliability of long time numerical integrations of chaotic systems [18]. The definition of R_F obviously implies that it can be used also as a *global* indicator, able to give a quantitative measure of the overall degree of stochasticity at a fixed value of energy: a phase-averaged *local* indicator is a *global* one, instead a global indicator is, in general, *non-local*. Perplexities which can be raised by the apparently cumbersome definition of R_F are answered on the light of the *pathologies* affecting the HH hamiltonian, which amount mainly to the degeneracies of its integrable limit [19, p.46]. Indeed, for most two dimensional DS's, more *naïve* Finsler geometric indicators suffice to discriminate between chaotic and regular orbits. Although the outcomes here presented (and in [2,9]) clearly support the reliability of Finsler GDA, nevertheless a better understanding is still waiting, as we need to test and extend the proposed criterion to more general and higher dimensional DS's. Moreover it is desirable to improve the rather phenomenological interpretation of R_F and possibly to predict theoretically the *threshold* above which stochasticity occurs. This goal amounts essentially to understand whether high R_F values either follow from or are a cause of stochastic behaviour (or both). The results obtained support the conviction [5–8,2] that negative curvature is unnecessary (if not irrelevant) to explain the onset of Chaos in realistic DS's, keeping however in mind that some faded global correlations exist. Instead, a separate spot concerns the long-lasting claims about the implications of scalar curvature: except for two dimensional or isotropic manifolds, it appears to have nothing to do with the behaviour of geodesics representing realistic DS's [8] and an hopefully coherent explanation of its irrelevance will be presented elsewhere. Among the issues still open, it is worth to investigate more deeply the relevance of the rotation of eigendirections of $\boldsymbol{\mathcal{H}}$: a direct inspection of the JLC equations seems to indicate that the rotation of the perturbation vector should give a contribution to instability, nevertheless a *separation by hand* of *interacting effects* often causes serious inconsistencies.

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[9] Cipriani P. and Di Bari M., Phys. Rev. E, **XX**, xxx (1998) *(to be submitted).* Our present work is devoted to a clarification of the issues here addressed and to a suitable extension of the method proposed. We plan to present the relevant outcomes in forthcoming papers.

[10] One of the outstanding outcomes of Pettini and coworkers investigations resides in the possibility of finding indicators of Chaos through *microcanonical averages*, without integrations of trajectories. It is an obvious price to pay for this achievement to renounce, *in principle*, to study the local behaviour of the DS, looking only to its average global features. The indicator found in [7] fails indeed to distinguish between regular and stochastic orbits on a given energy surface: it assumes in chaotic regions of the phase space a value in between those taken in two (disjoint) regular islands [9]. This is confirmed in [2] where a thorough exploration of the HH phase space reveals somewhere quantitative (and even qualitative) disagreements between dynamical behaviour and Riemannian description.

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[15] The non trivial effects consequent to an arbitrary rescaling of the time variable are well known within a general relativistic context [2,13] but should be considered also in Classical Mechanics.

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FIG. 1. Phase space portraits of HH system at $E = 1/8$: **a)** PSS obtained numerically integrating a sample of orbits up to T=10000 units; **b)** Gray-scale plot of $R^2_F(y_0, p_{y_0})$ computed following a set of geodesics for a lapse S=200 units. Darker dots represent smaller R_F values, the *White Stochastic Sea* corresponds to $R^2_{F_E} > 0.8$. The numbers label the initial conditions of figure 2.

FIG. 2. Values of $R^2_{F_E}$ for geodesics starting from i.c. "topologically equivalent" to those indicated on figure 1 for $E = 1/8$, at energies from $E = 0.095$ up to $E = 0.166$.

Fig.2